Q: Suppose a sphere has radius r and is centered at origin. what's its volume?

Sol1:

$$x^2 + y^2 + z^2 = \alpha^2$$
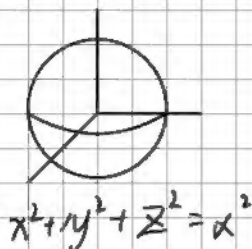
we seek Vol(S) as a double integral

For Height function: $z^2 = \alpha^2 - x^2 - y^2$

So upper hemisphere is $z = \sqrt{\alpha^2 - x^2 - y^2}$
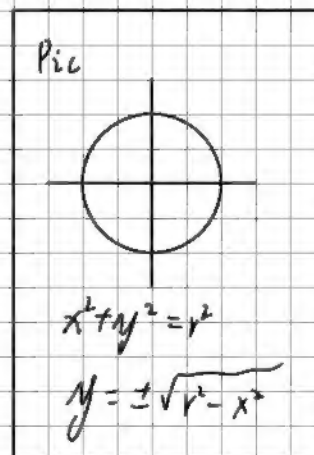
So lower hemisphere is $z = -\sqrt{\alpha^2 - x^2 - y^2}$

total height = upper - lower

$$h(x,y) = \sqrt{\alpha^2 - x^2 - y^2} - \left(-\sqrt{\alpha^2 - x^2 - y^2}\right)$$

$$= 2\sqrt{\alpha^2 - x^2 - y^2}$$

To parameterize R_r, we wrote

$$R_r = \left\{ (x,y) : x^2 + y^2 \leq \alpha^2 \right\}$$

$$= \left\{ (x,y) : -\alpha \leq x \leq \alpha, -\sqrt{\alpha^2 - x^2} \leq y \leq \sqrt{\alpha^2 - x^2} \right\}$$

$$\therefore Vol(s) = \iint_R h(x,y) \, dA$$

$$= \int_{x=-\alpha}^{\alpha} \int_{-\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}} \left(2\sqrt{\alpha^2 - x^2 - y^2} \right) dy \, dx$$

Pic

$$x^2 + y^2 = r^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

Inner Integral:

$$\int 2 \left[(\alpha^2 - x^2) - y^2\right]^{\frac{1}{2}} dy$$

$$= 2 \int \sqrt{\alpha^2 - x^2}\cos\theta \cdot \sqrt{\alpha^2 - x^2}\cos\theta \, d\theta$$

$$= 2(\alpha^2 - x^2) \int \cos^2\theta \, d\theta$$

$$= (\alpha^2 - x^2) \cdot \int 1 - \cos(2\theta) d\theta$$

$$= (\alpha^2 - x^2) \cdot (\theta + \tfrac{1}{2}\sin(2\theta)) + C$$

$$= (\alpha^2 - x^2) \cdot (\theta + \sin\theta\cos\theta) + C$$

$$= (\alpha^2 - x^2) \cdot \left(\sin^{-1}\left(\frac{y}{\sqrt{\alpha^2 - x^2}}\right) + \frac{y}{\sqrt{\alpha^2 - x^2}} \cdot \frac{\sqrt{\alpha^2 - x^2 - y^2}}{\sqrt{\alpha^2 - x^2}}\right) + C$$

$$= (\alpha^2 - x^2) \cdot \left(\sin^{-1}\left(\frac{y}{\sqrt{\alpha^2 - x^2}}\right) + y \cdot \sqrt{\alpha^2 - x^2 - y^2}\right) + C$$

$$\frac{y}{\sqrt{\alpha^2 - x^2}} = \sin\theta$$

$$y = \sin\theta \cdot \sqrt{\alpha^2 - x^2}$$

$$\sqrt{r^2 - x^2 - y^2} = \sqrt{\alpha^2 - x^2}\cos\theta$$

$$dy = \sqrt{\alpha^2 - x^2}\cos\theta \, d\theta$$

$\therefore$ Evaluating we obtain

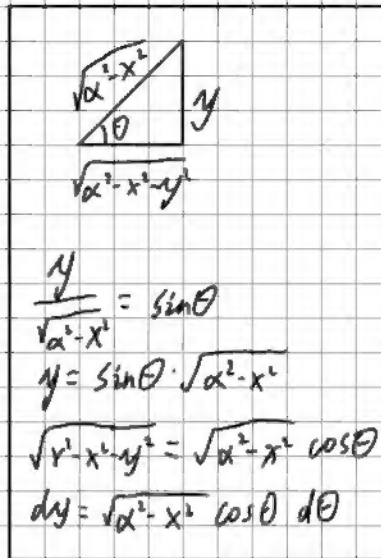$$\int_{y = -\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}} 2\sqrt{\alpha^2 - x^2 - y^2} \, dy$$

$$= \left[(\alpha^2 - x^2) \cdot \left(\sin^{-1}\left(\frac{y}{\sqrt{\alpha^2 - x^2}}\right) + y \cdot \sqrt{\alpha^2 - x^2 - y^2}\right) + C\right]_{-\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}}$$

$$= \left[(\alpha^2 - x^2) \cdot \sin^{-1}(1) + \sqrt{\alpha^2 - x^2} \cdot \sqrt{0}\right] - \left[(\alpha^2 - x^2) \cdot \sin^{-1}(-1) + \sqrt{\alpha^2 - x^2} \cdot \sqrt{0}\right]$$

$$= (\alpha^2 - x^2) \cdot \left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= (\alpha^2 - x^2) \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$
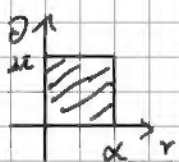
$$= \pi(\alpha^2 - x^2)$$

Outer integral

$$\int_{x=-\alpha}^{\alpha} \pi(\alpha^2 - x^2)\, dx$$

$$= \pi\left[r^2 x - \frac{1}{3} x^3 \right]_{-\alpha}^{\alpha}$$

$$= \pi\left[\left(\alpha^3 - \frac{1}{3}\alpha^3\right) - \left(-\alpha^3 + \frac{1}{3}\alpha^3\right) \right]$$

$$= \pi\left[2\alpha^3 - \frac{2}{3}\alpha^3 \right]$$

$$= \frac{4}{3}\pi\alpha^3$$

NB: That was computationally complicated

If we use polar coordinates for integral, the region and height function are much simpler

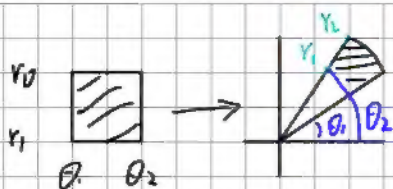$$R_{polar} = \left\{ (r, \theta),\ 0 \leq r \leq \alpha,\ 0 \leq \theta \leq 2\pi \right\}$$

$$h(x, y) = 2\sqrt{\alpha^2 - x^2 - y^2}$$

$$\Downarrow$$

$$h(r\cos\theta, r\sin\theta) = 2\sqrt{\alpha^2 - (r^2\cos^2\theta + r^2\sin^2\theta)} = 2\sqrt{\alpha^2 - r^2}$$

To consider the differential, consider a small rectangle In the cartesion plane corresponds to a circular plane

Area of a (small) circular section is:

$$\frac{1}{2} r_2^2 (\theta_2 - \theta_1) - \frac{1}{2} r_1^2 (\theta_2 - \theta_1)$$

$$= \frac{1}{2} (r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

$$= \frac{1}{2} (r_2 + r_1)(r_2 - r_1)(\theta_2 - \theta_1)$$

$$\therefore \Delta A = \frac{1}{2}(r_2 + r_1) \cdot \Delta r \cdot \Delta \theta = \frac{1}{2}(r_2 + r_1) \cdot \Delta A_{polar}$$

Now limiting as $\Delta A \to 0$ $(\Delta\theta \to 0, \Delta r \to 0)$

We see $\frac{1}{2}(r_2 + r_1) \longrightarrow \frac{1}{2} 2r^* = r^*$
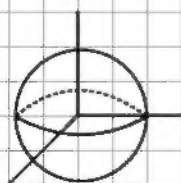
Hence in the limit we obtain

$$dA_{curt} = r \, dA_{polar}$$

Volumn of Sphere:

Sol 2: (with polar coordinates):

In polar coordinates (i.e. (r, θ, z) plane)

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{\alpha^2 - x^2 - y^2}$$
$$= \sqrt{\alpha^2 - r^2}$$

$$R_{polar} = [0, 2\pi] \times [0, \alpha]$$

$$Vol(S) = \iint_{R_{curve}} h(x,y) \, dA_{curve}$$

$$= \iint_{R_{polar}} h(r\cos\theta, r\sin\theta) \cdot r \, dA_{polar}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\alpha} 2\sqrt{\alpha^2 - r^2} \cdot r \, dr \, d\theta$$

Inner Integral:

$$\int_{r=0}^{r=\alpha} 2r\sqrt{\alpha^2 - r^2} \, dr \qquad\qquad u = a^2 - r^2$$
$$du = -2r \, dr$$

$$= \int_{r=0}^{\alpha} u^{\frac{1}{2}} \, du$$

$$= \left[-\frac{1}{3} u^{\frac{3}{2}} \right]_0^{\alpha}$$

$$= \left[-\frac{1}{3} (\alpha^2 - r^2)^{\frac{3}{2}} \right]_0^{\alpha}$$

$$= -\frac{2}{3}(\alpha^2 - \alpha^2)^{\frac{3}{2}} - \left(-\frac{2}{3}(\alpha^2 - 0)^{\frac{3}{2}} \right)$$

$$= \frac{2}{3}\alpha^3$$

Outer Integral:

$$\int_0^{2\pi} \frac{2}{3}\alpha^3 \, d\theta$$

$$= \frac{2}{3}\alpha^3 \int_0^{2\pi} d\theta$$

$$= \frac{2}{3}\alpha^3 \left[\theta \right]_0^{2\pi}$$

$$= \frac{4}{3}\pi\alpha^3$$

Ex. compute $\iint_R \sin(\sqrt{x^2+y^2})\, dA$ for the region

bounded by $x^2+y^2=1$, $x^2+y^2=9$

Sol: Turn the integral into polar form

$R_{polar} = \{(r,\theta) ; 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$

$f(x,y) = \sin(\sqrt{x^2+y^2})$ has polar form

$f(r\cos\theta, r\sin\theta) = \sin(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}) = \sin(r)$

$\therefore \iint_{R_{cart}} \sin(\sqrt{x^2+y^2})\, dA_{cart}$

$= \iint_{R_{polar}} \sin(r) \cdot r\, dA_{polar}$

$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} r\cdot\sin(r)\, dr\, d\theta$

$\begin{array}{ll} u=r & dv=\sin(r) \\ du=dr & v=-\cos(r) \end{array}$

$= \int_0^{2\pi} \left[-r\cos(r) - \int -\cos(r)\, dr\right]_1^3 d\theta$

$= \int_0^{2\pi} \left[-r\cos(r) + \sin(r)\right]_1^3 d\theta$

$= \int_0^{2\pi} (-3\cos(3) + \sin(3)) - (-\cos(1) + \sin(1))\, d\theta$

$= (\sin(3) - \sin(1) - 3\cos(3) + \cos(1)) \cdot [\theta]_0^{2\pi}$

$= 2\pi \cdot (\sin(3) - \sin(1) - 3\cos(3) + \cos(1))$

Exercise: Compute $\iint_R x\cdot\exp(-x^2-y^2)\, dA$ on R the disk of radius 3 about the origin.